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The stability of a superfluid rotating jet

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Abstract
We consider the linear stability of a cylindrical rotating jet of pure superfluid held together by surface tension. A necessary and sufficient condition for stability to axisymmetric disturbances is derived which corresponds to that of a classical inviscid fluid. For axisymmetric disturbances we find that the vortex tension does not affect the range of unstable axial wave numbers, only the temporal growth rate and the most critical wave number. A sufficient condition for the stability of a general disturbance is derived which corresponds to that of a classical inviscid fluid. We find for non-axisymmetric disturbances, that the vortex tension increases the range of unstable wave numbers. The temporal growth rates of the unstable azimuthal modes increase with vortex tension.

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1. Background and motivation

Helium is the only substance that remains liquid at the temperature of absolute zero at atmospheric pressure. At zero Kelvin helium is entirely superfluid, that is to say it is inviscid and flows without any viscous dissipation. At finite non-zero temperatures a fraction of helium becomes viscous, but as this fraction is negligible at temperatures below 1 K which are easily attained in the laboratory, we shall be concerned only with the motion of the superfluid. What makes the dynamics of the superfluid particularly interesting is the existence of vortex filaments. The key property of a superfluid vortex filament is the quantization of the circulation [1],

$$\oint_C \boldsymbol{v} \cdot \mathbf{d}\boldsymbol{l} = \Gamma \tag{1}$$

where $\boldsymbol{v}$ is the superfluid velocity field, $\Gamma = 9.97\ \mathrm{cm}^2\ \mathrm{s}^{-1}$ is the quantum of circulation (the ratio of Plank's constant and the mass of one helium atom) and C is an arbitrary integration path around the axis of the filament. Superfluid vortex filaments appear spontaneously when helium is rotated. It is found that $N = 2\Omega/\Gamma$ vortices per unit area thread the liquid, all aligned along the direction of rotation $\boldsymbol{\Omega}$.

Quantum effects are important only within the vortex core radius $a_0 \approx 10^{-8}$ cm. The smallness of this region and the fact that, in typical laboratory conditions ($0 < \Omega < 10$ rad s^{-1}) N is rather large, motivated the development of a hydrodynamic model of superfluidity [2, 3] often referred to as the HVBK model. The basic idea of this model is that a fluid particle contains a large number of vortices; hence superfluid vorticity (which is discrete in nature) is approximated as a continuum. The model was used with success [4–6] to explain transitions which were observed [7] in the Taylor–Couette flow of helium at finite temperatures.

Our work is motivated by the renewed interest [8] in some fundamental aspects of the HVBK model and by recent experiments with magnetically levitated superfluid drops [9, 10]. The experiments indicate that free surface configurations (e.g., drops and jets) are ideal testing grounds for the principles of superfluid hydrodynamics because they can be easily put into oscillation or rotation. The aim of this paper is to investigate the stability of a rotating superfluid jet, a simple geometry which has not been studied yet. In particular, we want to identify the difference between the motion of a classical inviscid Euler jet and the motion of a pure superfluid jet.

2. Model

The governing HVBK equations at zero temperature are

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nu_s \boldsymbol{\omega} \times (\nabla \times \widehat{\boldsymbol{\omega}}) \tag{2}$$

$$\nabla\cdot\boldsymbol{v} = 0 \tag{3}$$

where $\rho \approx 0.145$ g cm^{-3} is the superfluid density, p is the pressure, $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$ is the vorticity and $\widehat{\boldsymbol{\omega}} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$ is the unit vector in the direction of vorticity. The vortex tension parameter $\nu_s = (\Gamma/4\pi)\log(b_0/a_0)$ has the same dimension as kinematic viscosity but physically it is very different: it represents the ability of a vortex line to oscillate due to vortex waves which can be excited on the vortex lines themselves. $b_0 = (|\boldsymbol{\omega}|/\Gamma)^{-1/2}$ is the intervortex spacing.

The basic unperturbed state that we wish to investigate is that of an infinitely long, incompressible cylindrical column of superfluid helium, of radius a rotating rigidly at constant angular velocity Ω, about its axis, surrounded by a vacuum.

We non-dimensionalize the equations and boundary conditions taking the length scale to be the radius of the unperturbed jet a, and natural time scale $(\rho a^3/\gamma)^{1/2}$, where $\gamma = 0.35$ dyne cm^{-1} [11] is the surface tension parameter. This yields two dimensionless parameters,

$$\beta = \left(\frac{\rho}{a\gamma}\right)^{1/2}\nu_s \tag{4}$$

which gives a non-dimensional measure of the vortex tension parameter and the non-dimensional angular velocity, ω, defined by

$$\omega = \left(\frac{\rho a^3}{\gamma}\right)^{1/2}\Omega. \tag{5}$$

Henceforth, all quantities will be non-dimensional unless stated otherwise.

3. Linear stability analysis

We introduce cylindrical polar coordinates (r, ϕ, z) with the z-axis aligned along the axis of the cylindrical jet. In the unperturbed state, the free boundary is given by $r = 1$ and the velocity takes the form $\boldsymbol{v}_o = (0, \omega r, 0)$ in a frame of reference such that the net axial velocity is zero. The pressure inside the jet is given by $p_o = 1 + \omega^2(r^2 - 1)/2$.

In the perturbed state we denote the shape of the free boundary as $r = 1 + \eta(\phi, z, t)$ and look for solutions $\boldsymbol{u}, P$, where

$$\boldsymbol{v}(r, \phi, z, t) = \boldsymbol{v}_o + \boldsymbol{u}(r, \phi, z, t) \tag{6}$$

$$p(r, \phi, z, t) = p_o + P(r, \phi, z, t). \tag{7}$$

Substituting these expressions into the governing equations (2) and (3), assuming the perturbation quantities $\boldsymbol{u} = (u_r, u_\phi, u_z)$ to be small, we obtain the following linearized equations:

$$\frac{\partial u_r}{\partial t} + \omega\frac{\partial u_r}{\partial \phi} - 2\omega u_\phi = -\frac{\partial P}{\partial r} - \beta\left(\frac{\partial^2 u_\phi}{\partial z^2} - \frac{1}{r}\frac{\partial^2 u_z}{\partial \phi \partial z}\right) \tag{8}$$

$$\frac{\partial u_\phi}{\partial t} + \omega\frac{\partial u_\phi}{\partial \phi} + 2\omega u_r = -\frac{1}{r}\frac{\partial P}{\partial \phi} - \beta\left(\frac{\partial^2 u_z}{\partial r \partial z} - \frac{\partial^2 u_\phi}{\partial z^2}\right) \tag{9}$$

$$\frac{\partial u_z}{\partial t} + \omega\frac{\partial u_z}{\partial \phi} = -\frac{\partial P}{\partial z} \tag{10}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0. \tag{11}$$

In order to solve the system we need boundary conditions. These take the form of a kinematic and dynamic condition on the free surface. The kinematic condition, which ensures that fluid particles on the surface remain on the surface, may be expressed as

$$\frac{\mathrm{D}}{\mathrm{D}t}(r - 1 - \eta(\phi, z, t)) = 0 \qquad \text{on} \quad r = 1 + \eta(\phi, z, t) \tag{12}$$

which may be linearized to give the condition

$$u_r = \frac{\partial \eta}{\partial t} + \omega\frac{\partial \eta}{\partial \phi} \qquad \text{on} \quad r = 1. \tag{13}$$

The dynamic boundary condition ensures that at any point on the surface the pressure must be balanced by the normal force due to surface tension. In dimensional form this is given by $\gamma(1/R_1 + 1/R_2)$, where R_1 and R_2 are the principal radii of curvature of the perturbed free boundary. Substituting the perturbed solutions into this force balance and linearizing give the condition

$$P = -\omega^2\eta - \eta - \frac{\partial^2 \eta}{\partial \phi^2} - \frac{\partial^2 \eta}{\partial z^2} \qquad \text{on} \quad r = 1. \tag{14}$$

We seek a solution for $\boldsymbol{u}, P$ and η by applying normal modes in the form

$$\boldsymbol{u} = (\widehat{u}_r(r), \widehat{u}_\phi(r), \widehat{u}_z(r)) \exp(\sigma t + \mathrm{i}(kz + m\phi)) \tag{15}$$

$$P = \widehat{p}(r) \exp(\sigma t + \mathrm{i}(kz + m\phi)) \tag{16}$$

$$\eta = \widehat{\eta} \exp(\sigma t + \mathrm{i}(kz + m\phi)) \tag{17}$$

where the axial wave number $k \geqslant 0$ and azimuthal wave number $m = 0, 1, 2, \ldots$. In general the temporal exponent $\sigma = \sigma_1 + \mathrm{i}\sigma_2$ is complex and is determined in terms of the other parameters of the system. The problem is linearly stable if $\sigma_1 \leqslant 0$ and linearly unstable if $\sigma_1 > 0$.

Substituting these forms of the solutions into (2) and (3) gives

$$su_r - 2\omega u_\phi = -\frac{\mathrm{d}P}{\mathrm{d}r} - \beta\left(-k^2 u_\phi + \frac{mk}{r}u_z\right) \tag{18}$$

$$su_\phi + 2\omega u_r = -\frac{\mathrm{i}mP}{r} - \beta\left(\mathrm{i}k\frac{\mathrm{d}u_z}{\mathrm{d}r} + k^2 u_r\right) \tag{19}$$

$$su_z = -\mathrm{i}kP \tag{20}$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru_r) + \frac{\mathrm{i}m}{r}u_\phi + \mathrm{i}ku_z = 0 \tag{21}$$

and substituting into the boundary conditions (13) and (14) gives

$$u_r(1) = s\eta \tag{22}$$

$$P(1) = -(1 + \omega^2 - m^2 - k^2)\eta \tag{23}$$

where $s = \sigma + \mathrm{i}m\omega$ and the hats have been omitted for clarity.

Eliminating u_r, u_ϕ, u_z from the above equations yields the following equation for P:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}P}{\mathrm{d}r}\right) - \frac{m^2}{r^2}P - l^2 P = 0 \tag{24}$$

where

$$l^2 = \frac{(s^2 + (2\omega + \beta k^2)^2)k^2}{s^2 + \beta k^2(2\omega + \beta k^2)}. \tag{25}$$

This has solution $P = AI_m(lr)$, where I_m is the modified Bessel function of the first kind and A is a constant. The corresponding term involving $K_m(lr)$ has been discarded as the pressure must be bounded at $r = 0$.

In order to apply the boundary conditions (13) and (14) we need to express u_r in terms of pressure. This can be found to be

$$u_r = -\frac{A}{r}\left\{\frac{2\mathrm{i}m\omega}{s^2 + (2\omega + \beta k^2)^2}I_m(lr) + \frac{k^2 r}{ls}I_m'(lr)\right\}. \tag{26}$$

Applying the boundary conditions yields the following eigenvalue problem:

$$s^2 = \left\{\frac{k^2}{l}\frac{I_m'(l)}{I_m(l)} + \frac{2\mathrm{i}m\omega s}{s^2 + (2\omega + \beta k^2)^2}\right\}(1 + \omega^2 - m^2 - k^2) \tag{27}$$

with l as in (25) and $I_m'(l) = \mathrm{d}I_m(l)/\mathrm{d}l$.

4. Planar disturbances ($k = 0$)

In the case when $k = 0$, when the disturbances are restricted to planes perpendicular to the axis of the cylinder, one can see from equations (18)–(21) that the vortex tension effects are no longer present in the equations. Thus, the resulting problem reduces to that of planar disturbances of an inviscid rotating jet. The stability criterion for this disturbance was first considered by Hocking and Michael [12] who showed that the condition for stability is

$$\omega^2 \leqslant m(m + 1).$$

5. Non-planar disturbances ($k \neq 0$)

5.1. Axisymmetric perturbations

We shall start by considering axisymmetric disturbances. Taking $m = 0$ the problem reduces to solving

$$s^2 = k^2 C(l)(1 + \omega^2 - k^2) \tag{28}$$

where $C(l) = l^{-1} I_1(l)/I_0(l)$ together with equation (25) which contains the vortex tension parameter β. We have shown in the appendix that a sufficient condition for stability of the jet to axisymmetrical disturbances is $k^2 \geqslant 1 + \omega^2$. We shall demonstrate that this is also a necessary condition.

In the appendix we have shown that s^2 is real for axisymmetric disturbances; thus stability will only occur if $s^2 \leqslant 0$. From (25) we see that l^2 must also be real and rearranging we find

$$s^2 = \frac{(2\omega + \beta k^2 - \beta l^2)}{l^2 - k^2}(2\omega + \beta k^2)k^2. \tag{29}$$

Substituting into (28) gives

$$\frac{1}{C(l)} = X\left(\frac{l^2 - k^2}{\beta l^2 - \beta k^2 - 2\omega}\right) \tag{30}$$

where

$$X = -\frac{(1 + \omega^2 - k^2)}{2\omega + \beta k^2}. \tag{31}$$

From (29) we see that stable conditions only occur when all the roots of (30) are purely imaginary or are real with either $|l| < k$ or $|l| \geqslant \sqrt{k^2 + 2\omega/\beta}$. The magnitude of the real roots may be found by considering the intersections of the graphs of $1/C(l)$ and $X(l^2 - k^2)/(\beta l^2 - \beta k^2 - 2\omega)$. The first graph is an even function of l which is monotonically increasing for positive l which has a minimum value of 2 when $l = 0$ and is of order l for large values of l. The roots occur in the ranges $|l| \geqslant \sqrt{k^2 + 2\omega/\beta}$ and $|l| < k$ if $X \geqslant 0$ and in the range $k < l < \sqrt{k^2 + 2\omega/\beta}$ for $X < 0$. The column is therefore stable to axisymmetrical disturbances unless $X < 0$ or the column is only stable when $k \geqslant \sqrt{1 + \omega^2}$. Thus unstable disturbances exist for axial wave numbers in the range $0 < k \leqslant \sqrt{1 + \omega^2}$. This corresponds to the findings of Hocking [13] for the case without vortex tension. Thus the presence of vortex lines does not alter the range of unstable axial wave numbers.

To compute the growth rates of the unstable disturbances, we use a Newton–Raphson method to converge to real values of s and l for $0 < k \leqslant \sqrt{1 + \omega^2}$ given values of the parameters β and ω. Plots of the unstable growth rate σ_1 against axial wave number k for increasing values of β are plotted in figures 1 and 2 for angular velocities $\omega = 1$ and $\omega = 5$, respectively. Figures 1 and 2 show that while leaving the range of unstable axial wave numbers unaltered, the effect of increasing the vortex tension is to increase the temporal growth rate up to a limiting form which occurs as $\beta \to \infty$. Thus the vortex tension acts in direct contrast to the effect of fluid viscosity [14].

Considering the case of infinite β in particular, we see from equation (25) that $l \to k$ as $\beta \to \infty$. Therefore the eigenvalue problem reduces to

$$s^2 = k^2 C(k)(1 + \omega^2 - k^2).$$

This is similar to the eigenvalue problem that results from considering the non-rotating inviscid jet, with the 1 replaced by $1 + \omega^2$ [15].

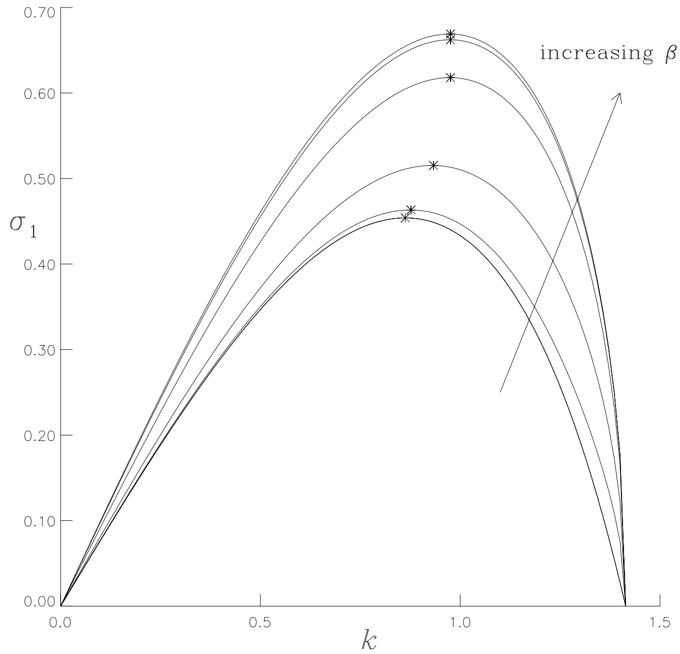

Figure 1. Values of σ_1 for the unstable axisymmetric modes rotating with angular velocity $\omega = 1$, plotted as a function of k for increasing values of $\beta = 0$, 0.01, 0.1, 1, 10, ∞ as indicated by the arrow. The asterisks represent the value of $k = k_{\max}$ at which σ_1 is maximum.

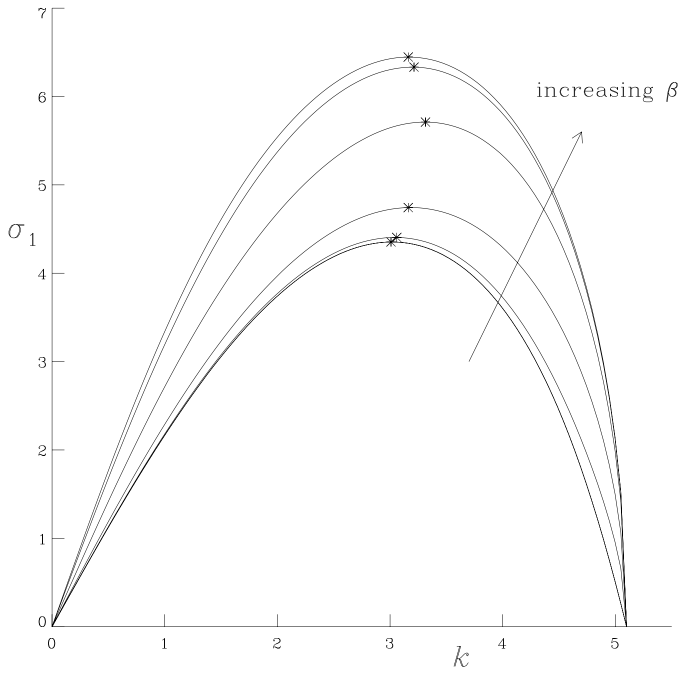

Figure 2. Values of σ_1 for the unstable axisymmetric modes rotating with angular velocity $\omega = 5$, plotted as a function of k for increasing values of $\beta = 0$, 0.01, 0.1, 1, 10, ∞ as indicated by the arrow. The asterisks represent the value of $k = k_{\max}$ at which σ_1 is maximum.

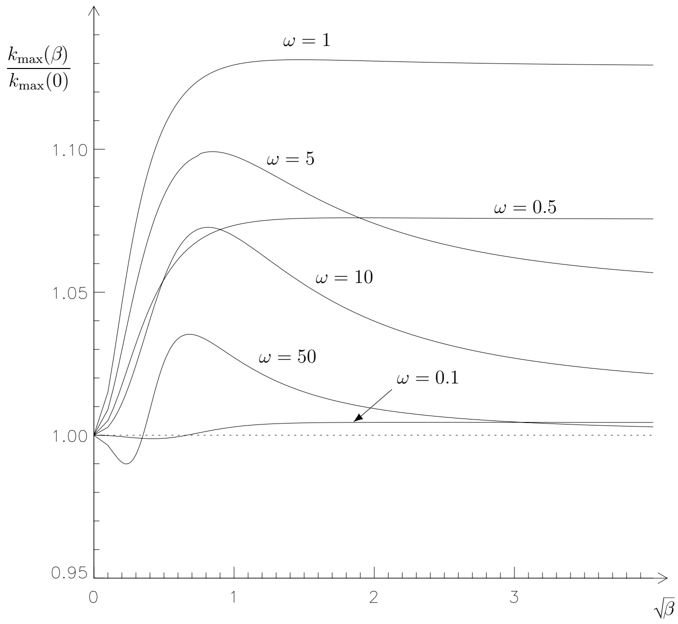

Figure 3. Values of $k_{\max}(\beta)/k_{\max}(0)$ plotted against $\sqrt{\beta}$ for various values of ω. The dotted line corresponds to $\omega = 0$.

Another interesting finding comes from looking at the axial wave number $k_{\max}$ at which the value of σ_1 is greatest, these points are indicated by asterisks in figures 1 and 2. For a given value of ω and β this will give the most unstable wavelength. Considering figure 1 we see that $k_{\max}$ increases monotonically with β, that is for a superfluid jet rotating with angular velocity $\omega = 1$, the most dominant wave number increases with vortex tension and is always greater than that for an inviscid jet. Considering figure 2, in which the jet is rotating with angular velocity $\omega = 5$, we see that the most unstable wave number of a superfluid jet is always greater than that of an inviscid jet as in the previous example. However, the increase is no longer monotonic. As the vortex tension increases $k_{\max}$ increases to a maximum value and then decreases slightly as $\beta \to \infty$.

This effect is illustrated more clearly in figure 3 in which the relative value of $k_{\max}$ with respect to that of a classical inviscid jet, is plotted against $\sqrt{\beta}$ for various values of ω. One can see that at moderate angular velocities ($\omega = 0.5$ to 10) the critical axial wave number of the most unstable mode increases initially with vortex tension; however, for the higher angular velocities in this range $k_{\max}$ reaches a maximum and decreases as $\beta \to \infty$. For large values of ω ($\omega = 50$) the critical wave number relative to a classical inviscid jet decreases initially before increasing to a level higher than for a classical jet. A similar effect is noticed at smaller angular velocities ($\omega = 0.1$), though the effect is less pronounced. From figure 3 we see that for certain values of ω and β there is a marked difference between the critical axial wave number at which a superfluid jet with and without vortices becomes unstable (an increase of 13% for $\omega = 1$ and $\beta = 1$).

5.2. Non-axisymmetric perturbations

We shall now turn our attention to the non-axisymmetric modes. In this case s^2 is no longer always real and no necessary condition for linear stability has been obtained. However,

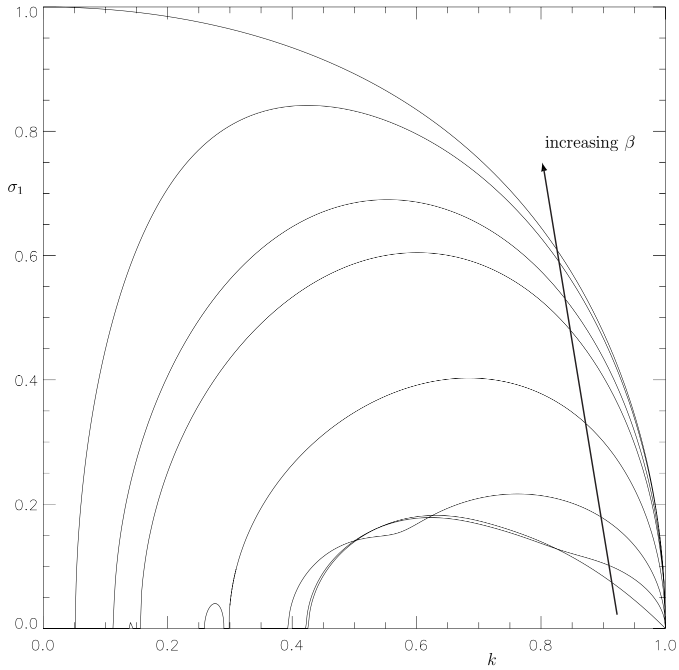

Figure 4. Values of σ_1 for the $m = 1$ mode rotating with angular velocity $\omega = 1$, plotted as a function of k for $\beta = 0, 0.01, 0.1, 1, 5, 10, 50, \infty$.

following Pedley [16] we have proved in the appendix that a sufficient condition for linear stability is

$$1 + \omega^2 - m^2 - k^2 \leqslant 0. \tag{32}$$

For given values of the wave numbers m, k and parameters ω, β equations (25) and (27) were solved using Newton–Raphson iteration on the real and imaginary parts of s and l, respectively. The mode is unstable provided $\mathrm{Re}(s) \neq 0$ since if (s, l) is a solution then so is $(-s^*, l^*)$ where $*$ represents the complex conjugate.

Results for the $m = 1$ mode at angular velocity $\omega = 1$ are presented in figure 4 for various values of β. From (32) only the $m = 0$ and $m = 1$ mode may be unstable for rotation rates up to $\omega = \sqrt{3}$. One can see that as β increases the range of unstable axial wave numbers increases together with the maximum value of σ_1. Comparing with the results from the axisymmetric case (figure 1) one can see that for small values of β (up to about 5) the axisymmetric mode is the most dominant; however, as β increases the $m = 1$ mode becomes the most dominant and $k_{\mathrm{max}} \to 0$ as $\beta \to \infty$.

At larger rotation rates the picture is more complicated as more of the azimuthal modes become unstable. Results for azimuthal modes $m = 0, 1, \ldots, 5$ at angular velocity $\omega = 5$ are presented in figure 5 for $\beta = 0.1$. In this case, as for a classical jet, the maximum growth rate is $\sigma_1 = \sqrt{26}$ which occurs for the $m = 3$ mode at $k_{\mathrm{max}} = 0$. The growth rates of each mode at $k = 0$ can be found analytically by taking the limit as $k \to 0$ of equations (25) and (27) keeping β finite. In this case we find that

$$s(k = 0) = \mathrm{i}\omega \pm \sqrt{(m - 1)(\omega^2 - m(m + 1))}. \tag{33}$$

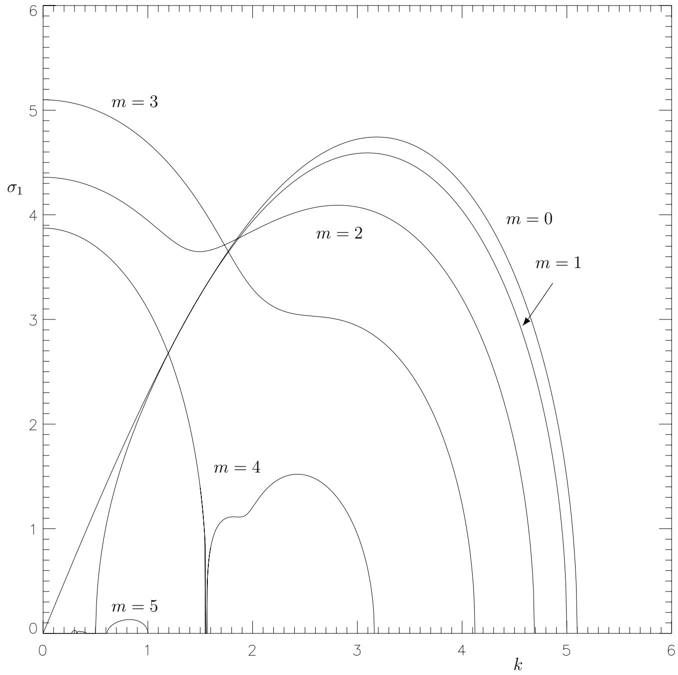

Figure 5. Values of σ_1 plotted against k for the azimuthal modes $m = 0, 1, \ldots, 5$ at $\omega = 5$.

Thus provided $m \geqslant 1$ and $\omega^2 \geqslant m(m+1)$, $\mathrm{Re}(s) \neq 0$ and the mode is unstable at $k = 0$. It is interesting to note that (33) is independent of the vortex tension parameter β. As β increases the maximum value of σ_1 remains at $\sqrt{26}$ until $\beta \approx 5$ at which the axisymmetric mode becomes most dominant. As β increases further the $m = 1$, $m = 2$ and $m = 3$ in turn become the most dominant mode.

6. Discussion

We have identified two governing dimensionless parameters β and ω, which describe, respectively, the intensity of the vortex tension and the rotation, and studied the linear stability of a rotating superfluid jet. It is a non-zero value of β which distinguishes a superfluid jet from a classical inviscid jet, as β is proportional to the quantum of circulation and hence to Plank's constant. Our analysis shows that the stability of a superfluid jet is different to that of a classical jet only as far as non-planar ($k \neq 0$) infinitesimal perturbations are concerned. In the case of axisymmetric disturbances ($m = 0$) the range of unstable axial wave numbers k is the same as in the classical case. What is different is the exponential growth, which is faster the larger β is. Non-axisymmetric disturbances ($m \neq 0$) become important only at relatively large values of β and tend to destabilize perturbations in the limit of long wavelengths ($k \to 0$). Inequality (32) shows that the number of potentially unstable azimuthal modes increases as the angular velocity increases.

In principle, using jets with different values of β and different rotation rates ω, these different types of behaviour can be observed. In practice, however, it is difficult to produce jets with β sufficiently large for the stability curves to be different from the classical ones: at $a = 0.01$ cm at $\Omega = 100$ rad sec^{-1} we have only $\beta = 0.006$. We conclude that the quantization of the circulation has virtually no observable consequence on the stability of a rotating superfluid jet, which can thus be considered a remarkable and realistic example of Eulerian motion.

Appendix

Here we extend the methods of Pedley [16] in which he considered disturbances of a rotating column of inviscid fluid, to prove two results for a rotating column of superfluid.

We can use equations (18) and (21) to eliminate u_ϕ, u_z and p to give a differential equation for u_r, which takes the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{T}{r}\frac{\mathrm{d}(ru_r)}{\mathrm{d}r}\right) = \left[1 + \frac{2\omega T\theta k^2}{s^2 + 2\omega\beta\theta k^2} + \frac{2\mathrm{i}m\omega\theta r}{s}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{T}{r^2}\right)\right]u_r \tag{34}$$

where

$$T = \frac{r^2}{m^2 + k^2 r^2} \qquad \theta = 1 + \frac{\beta k^2}{2\omega}.$$

Expressed in terms of u_r the boundary condition at the free surface may be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}(ru_r) = \frac{2\mathrm{i}m\omega\theta u_r}{s} + \frac{(1 + \omega^2 - m^2 - k^2)u_r}{Ts^2} \tag{35}$$

evaluated at $r = 1$. Multiplying equation (34) by ru_r^* (where u_r^* represents the complex conjugate of u_r) and integrating over the radius of the jet and using the boundary condition (35) gives

$$\frac{2\mathrm{i}m\omega\theta T}{s}|u_r|^2 + (1 + \omega^2 - m^2 - k^2)\frac{|u_r|^2}{s^2} = A_1 + A_2 + \frac{4\omega^2 k^2\theta}{s^2 + 2\omega\beta k^2\theta}A_3 + \frac{2\mathrm{i}m\omega\theta}{s}A_4 \tag{36}$$

evaluated at $r = 1$. Assuming that a disturbance exists the integrals $A_1, \ldots, A_4$ are strictly positive and given by

$$A_1 = \int_0^1 \frac{T}{r}\left|\frac{\mathrm{d}}{\mathrm{d}r}(ru_r)\right|^2 \mathrm{d}r \qquad A_2 = \int_0^1 r|u_r|^2\,\mathrm{d}r$$

$$A_3 = \int_0^1 rT|u_r|^2\,\mathrm{d}r \qquad A_4 = \int_0^1 r^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{T}{r^2}\right)|u_r|^2\,\mathrm{d}r.$$

We can use equation (36) to prove two useful results.

Result 1. *A sufficient condition for the linear stability of a general disturbance with axial wave number k and azimuthal wave number m is that*

$$1 + \omega^2 - m^2 - k^2 \leqslant 0.$$

Taking the real part of s times equation (36) we find

$$\mathrm{Re}(s)\left[A_1 + A_2 + \frac{4\omega^2 k^2\theta(|s|^2 + 2\omega\beta k^2\theta)}{|s^2 + 2\omega\beta k^2\theta|^2}A_3 - \frac{(1 + \omega^2 - m^2 - k^2)}{|s|^2}|u_r|^2\right] = 0.$$

Unstable disturbances with $\mathrm{Re}(s) \neq 0$ are impossible if the bracketed term is strictly positive. Thus a sufficient condition for stability is

$$1 + \omega^2 - m^2 - k^2 \leqslant 0$$

Result 2. *For axisymmetric disturbances s^2 is real.*

For axisymmetric disturbances $m = 0$ and taking the imaginary part of s^2 times equation (36) we find

$$\mathrm{Im}(s^2)\left[A_1 + A_2 + \frac{4\omega^2\beta k^4\theta}{|s^2 + 2\omega\beta k^2\theta|^2}A_3\right] = 0.$$

The bracketed term is strictly positive, thus the imaginary part of s^2 is zero and therefore s^2 is entirely real for axisymmetric disturbances.

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